Later we will need precise language to discuss the notion of one real number being "close to" another. If a is a given real number, then saying that a real number x is "close to" a should mean that the distance $|x - a|$ between them is "small." A context in which this idea can be discussed is provided by the terminology of neighborhoods, which we now define.

2.2.7 Definition Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the **ε-neighborhood** of a is the set $V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}$.

For $a \in \mathbb{R}$, the statement that x belongs to $V_\varepsilon(a)$ is equivalent to either of the statements (see Figure 2.2.4)

$$-\varepsilon < x - a < \varepsilon \iff a - \varepsilon < x < a + \varepsilon.$$

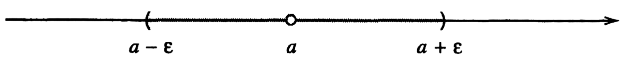

Figure 2.2.4 An ε-neighborhood of a

2.2.8 Theorem *Let $a \in \mathbb{R}$. If x belongs to the neighborhood $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.*

Proof. If a particular x satisfies $|x - a| < \varepsilon$ for every $\varepsilon > 0$, then it follows from 2.1.9 that $|x - a| = 0$, and hence $x = a$. Q.E.D.

2.2.9 Examples **(a)** Let $U := \{x : 0 < x < 1\}$. If $a \in U$, then let ε be the smaller of the two numbers a and $1 - a$. Then it is an exercise to show that $V_\varepsilon(a)$ is contained in U. Thus each element of U has some ε-neighborhood of it contained in U.

(b) If $I := \{x : 0 \le x \le 1\}$, then for any $\varepsilon > 0$, the ε-neighborhood $V_\varepsilon(0)$ of 0 contains points not in I, and so $V_\varepsilon(0)$ is not contained in I. For example, the number $x_\varepsilon := -\varepsilon/2$ is in $V_\varepsilon(0)$ but not in I.

(c) If $|x - a| < \varepsilon$ and $|y - b| < \varepsilon$, then the Triangle Inequality implies that

$$\begin{aligned} |(x + y) - (a + b)| &= |(x - a) + (y - b)| \\ &\le |x - a| + |y - b| < 2\varepsilon. \end{aligned}$$

Thus if x, y belong to the ε-neighborhoods of a, b, respectively, then $x + y$ belongs to the 2ε-neighborhood of $a + b$ (but not necessarily to the ε-neighborhood of $a + b$). □

Exercises for Section 2.2

1. If $a, b \in \mathbb{R}$ and $b \neq 0$, show that:
   (a) $|a| = \sqrt{a^2}$,   (b) $|a/b| = |a|/|b|$.
2. If $a, b \in \mathbb{R}$, show that $|a + b| = |a| + |b|$ if and only if $ab \ge 0$.
3. If $x, y, z \in \mathbb{R}$ and $x \le z$, show that $x \le y \le z$ if and only if $|x - y| + |y - z| = |x - z|$. Interpret this geometrically.

4. Show that $|x - a| < \varepsilon$ if and only if $a - \varepsilon < x < a + \varepsilon$.
5. If $a < x < b$ and $a < y < b$, show that $|x - y| < b - a$. Interpret this geometrically.
6. Find all $x \in \mathbb{R}$ that satisfy the following inequalities:
   (a) $|4x - 5| \le 13$, (b) $|x^2 - 1| \le 3$.
7. Find all $x \in \mathbb{R}$ that satisfy the equation $|x + 1| + |x - 2| = 7$.
8. Find all values of x that satisfy the following equations:
   (a) $x + 1 = |2x - 1|$, (b) $2x - 1 = |x - 5|$.
9. Find all values of x that satisfy the following inequalities. Sketch graphs.
   (a) $|x - 2| \le x + 1$, (b) $3|x| \le 2 - x$.
10. Find all $x \in \mathbb{R}$ that satisfy the following inequalities.
   (a) $|x - 1| > |x + 1|$, (b) $|x| + |x + 1| < 2$.
11. Sketch the graph of the equation $y = |x| - |x - 1|$.
12. Find all $x \in \mathbb{R}$ that satisfy the inequality $4 < |x + 2| + |x - 1| < 5$.
13. Find all $x \in \mathbb{R}$ that satisfy both $|2x - 3| < 5$ and $|x + 1| > 2$ simultaneously.
14. Determine and sketch the set of pairs (x, y) in $\mathbb{R} \times \mathbb{R}$ that satisfy:
   (a) $|x| = |y|$, (b) $|x| + |y| = 1$,
   (c) $|xy| = 2$, (d) $|x| - |y| = 2$.
15. Determine and sketch the set of pairs (x, y) in $\mathbb{R} \times \mathbb{R}$ that satisfy:
   (a) $|x| \le |y|$, (b) $|x| + |y| \le 1$,
   (c) $|xy| \le 2$, (d) $|x| - |y| \ge 2$.
16. Let $\varepsilon > 0$ and $\delta > 0$, and $a \in \mathbb{R}$. Show that $V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ-neighborhoods of a for appropriate values of γ.
17. Show that if $a, b \in \mathbb{R}$, and $a \ne b$, then there exist ε-neighborhoods U of a and V of b such that $U \cap V = \emptyset$.
18. Show that if $a, b \in \mathbb{R}$ then
   (a) $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ and $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$.
   (b) $\min\{a, b, c\} = \min\{\min\{a, b\}, c\}$.
19. Show that if $a, b, c \in \mathbb{R}$, then the "middle number" is $\mathrm{mid}\{a, b, c\} = \min\{\max\{a, b\}, \max\{b, c\}, \max\{c, a\}\}$.

Section 2.3 The Completeness Property of $\mathbb{R}$

Thus far, we have discussed the algebraic properties and the order properties of the real number system $\mathbb{R}$. In this section we shall present one more property of $\mathbb{R}$ that is often called the "Completeness Property." The system $\mathbb{Q}$ of rational numbers also has the algebraic and order properties described in the preceding sections, but we have seen that $\sqrt{2}$ *cannot* be represented as a rational number; therefore $\sqrt{2}$ does not belong to $\mathbb{Q}$. This observation shows the necessity of an additional property to characterize the real number system. This additional property, the Completeness (or the Supremum) Property, is an essential property of $\mathbb{R}$, and we will say that $\mathbb{R}$ is a *complete ordered field*. It is this special property that permits us to define and develop the various limiting procedures that will be discussed in the chapters that follow.

There are several different ways to describe the Completeness Property. We choose to give what is probably the most efficient approach by assuming that each nonempty bounded subset of $\mathbb{R}$ has a supremum.

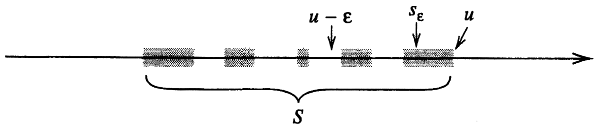

Figure 2.3.2 $u = \sup S$

2.3.5 Examples **(a)** If a nonempty set S_1 has a finite number of elements, then it can be shown that S_1 has a largest element u and a least element w. Then $u = \sup S_1$ and $w = \inf S_1$, and they are both members of S_1. (This is clear if S_1 has only one element, and it can be proved by induction on the number of elements in S_1; see Exercises 12 and 13.)

(b) The set $S_2 := \{x : 0 \leq x \leq 1\}$ clearly has 1 for an upper bound. We prove that 1 is its supremum as follows. If $v < 1$, there exists an element $s' \in S_2$ such that $v < s'$. (Name one such element s'.) Therefore v is not an upper bound of S_2 and, since v is an arbitrary number $v < 1$, we conclude that $\sup S_2 = 1$. It is similarly shown that $\inf S_2 = 0$. Note that both the supremum and the infimum of S_2 are contained in S_2.

(c) The set $S_3 := \{x : 0 < x < 1\}$ clearly has 1 for an upper bound. Using the same argument as given in (b), we see that $\sup S_3 = 1$. In this case, the set S_3 does *not* contain its supremum. Similarly, $\inf S_3 = 0$ is not contained in S_3. □

The Completeness Property of ℝ

It is not possible to prove on the basis of the field and order properties of ℝ that were discussed in Section 2.1 that every nonempty subset of ℝ that is bounded above has a supremum in ℝ. However, it is a deep and fundamental property of the real number system that this is indeed the case. We will make frequent and essential use of this property, especially in our discussion of limiting processes. The following statement concerning the existence of suprema is our final assumption about ℝ. Thus, we say that ℝ is a *complete ordered field.*

2.3.6 The Completeness Property of ℝ *Every nonempty set of real numbers that has an upper bound also has a supremum in ℝ.*

This property is also called the **Supremum Property** of ℝ. The analogous property for infima can be deduced from the Completeness Property as follows. Suppose that S is a nonempty subset of ℝ that is bounded below. Then the nonempty set $\bar{S} := \{-s : s \in S\}$ is bounded above, and the Supremum Property implies that $u := \sup \bar{S}$ exists in ℝ. The reader should verify in detail that $-u$ is the infimum of S.

Exercises for Section 2.3

1. Let $S_1 := \{x \in \mathbb{R} : x \geq 0\}$. Show in detail that the set S_1 has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.
2. Let $S_2 := \{x \in \mathbb{R} : x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.
3. Let $S_3 = \{1/n : n \in \mathbb{N})$. Show that $\sup S_3 = 1$ and $\inf S_3 \geq 0$. (It will follow from the Archimedean Property in Section 2.4 that $\inf S_3 = 0$.)
4. Let $S_4 := \{1 - (-1)^n/n : n \in \mathbb{N}\}$. Find $\inf S_4$ and $\sup S_4$.

5. Find the infimum and supremum, if they exist, of each of the following sets.
   (a) $A := \{x \in \mathbb{R} : 2x + 5 > 0\}$, (b) $B := \{x \in \mathbb{R} : x + 2 \geq x^2\}$,
   (c) $C := \{x \in \mathbb{R} : x < 1/x\}$, (d) $D := \{x \in \mathbb{R} : x^2 - 2x - 5 < 0\}$.
6. Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Prove that $\inf S = -\sup\{-s : s \in S\}$.
7. If a set $S \subseteq \mathbb{R}$ contains one of its upper bounds, show that this upper bound is the supremum of S.
8. Let $S \subseteq \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound of S if and only if the conditions $t \in \mathbb{R}$ and $t > u$ imply that $t \notin S$.
9. Let $S \subseteq \mathbb{R}$ be nonempty. Show that if $u = \sup S$, then for every number $n \in \mathbb{N}$ the number $u - 1/n$ is not an upper bound of S, but the number $u + 1/n$ is an upper bound of S. (The converse is also true; see Exercise 2.4.3.)
10. Show that if A and B are bounded subsets of $\mathbb{R}$, then $A \cup B$ is a bounded set. Show that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.
11. Let S be a bounded set in $\mathbb{R}$ and let S_0 be a nonempty subset of S. Show that $\inf S \leq \inf S_0 \leq \sup S_0 \leq \sup S$.
12. Let $S \subseteq \mathbb{R}$ and suppose that $s^* := \sup S$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.
13. Show that a nonempty finite set $S \subseteq \mathbb{R}$ contains its supremum. [*Hint:* Use Mathematical Induction and the preceding exercise.]
14. Let S be a set that is bounded below. Prove that a lower bound w of S is the infimum of S if and only if for any $\varepsilon > 0$ there exists $t \in S$ such that $t < w + \varepsilon$.

Section 2.4 Applications of the Supremum Property

We will now discuss how to work with suprema and infima. We will also give some very important applications of these concepts to derive fundamental properties of $\mathbb{R}$. We begin with examples that illustrate useful techniques in applying the ideas of supremum and infimum.

2.4.1 Examples (a) It is an important fact that taking suprema and infima of sets is compatible with the algebraic properties of $\mathbb{R}$. As an example, we present here the compatibility of taking suprema and addition.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a + S := \{a + s : s \in S\}$. We will prove that

$$\sup(a + S) = a + \sup S.$$

If we let $u := \sup S$, then $x \leq u$ for all $x \in S$, so that $a + x \leq a + u$. Therefore, $a + u$ is an upper bound for the set $a + S$; consequently, we have $\sup(a + S) \leq a + u$.

Now if v is *any* upper bound of the set $a + S$, then $a + x \leq v$ for all $x \in S$. Consequently $x \leq v - a$ for all $x \in S$, so that $v - a$ is an upper bound of S. Therefore, $u = \sup S \leq v - a$, which gives us $a + u \leq v$. Since v is any upper bound of $a + S$, we can replace v by $\sup(a + S)$ to get $a + u \leq \sup(a + S)$.

Combining these inequalities, we conclude that

$$\sup(a + S) = a + u = a + \sup S.$$

For similar relationships between the suprema and infima of sets and the operations of addition and multiplication, see the exercises.